Bearing

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Let us now calculate the bearing and distance from point A at latitude θ_A and longitude ϕ_A to point B at latitude θ_B and longitude ϕ_B. To accomplish this, we will make use of a space rectangular coordinate system fixed with the Earth such that the center of the Earth is at the origin, the London meridian intersects the positive x axis, the north pole lies on the positive z axis, and longitude meridian $+\frac{\pi}{2}$ intersects the positive y axis. From our understanding of the geographical coordinate system, if we let the radius of the Earth be unit distance, the coordinates of point $B(x_B, y_B, z_B)$ will be given by

$$x_B = \cos\theta_B \cos\phi_B \quad (1)$$

$$y_B = \cos\theta_B \sin\phi_B \quad (2)$$

$$z_B = \sin\theta_B \quad (3)$$

If we rotate the Earth about the z axis by $(-\phi_A)$, point A will now lie on the London meridian. Letting the updated coordinates of point B be (u, v, w), we have

$$\begin{aligned}
u &= x_B \cos(-\phi_A) - y_B \sin(-\phi_A) \\
u &= x_B \cos\phi_A + y_B \sin\phi_A \\
u &= \cos\theta_B \cos\phi_B \cos\phi_A + \cos\theta_B \sin\phi_B \sin\phi_A \\
u &= \cos\theta_B (\cos\phi_B \cos\phi_A + \sin\phi_B \sin\phi_A) \\
u &= \cos\theta_B \cos(\phi_B - \phi_A) \quad (4)
\end{aligned}$$

$$\begin{aligned}
v &= x_B \sin(-\phi_A) + y_B \cos(-\phi_A) \\
v &= y_B \cos\phi_A - x_B \sin\phi_A \\
v &= \cos\theta_B \sin\phi_B \cos\phi_A - \cos\theta_B \cos\phi_B \sin\phi_A \\
v &= \cos\theta_B (\sin\phi_B \cos\phi_A - \cos\phi_B \sin\phi_A) \\
v &= \cos\theta_B \sin(\phi_B - \phi_A) \quad (5)
\end{aligned}$$

$$\begin{aligned}
w &= z_B \\
w &= \sin\theta_B \quad (6)
\end{aligned}$$

Now, if we were to rotate about the y axis by $-(\frac{\pi}{2} - \theta_A) \equiv \theta_A - \frac{\pi}{2}$, point A will now be at the north pole. Letting the coordinates of point B now be (x, y, z), we have

$$\begin{aligned} z &= w\cos\left(\theta_A - \frac{\pi}{2}\right) - u\sin\left(\theta_A - \frac{\pi}{2}\right) \\ z &= w\sin\theta_A + u\cos\theta_A \\ z &= \sin\theta_A\sin\theta_B + \cos\theta_A\cos\theta_B\cos(\phi_B - \phi_A) \end{aligned} \tag{7}$$

$$\begin{aligned} x &= w\sin\left(\theta_A - \frac{\pi}{2}\right) + u\cos\left(\theta_A - \frac{\pi}{2}\right) \\ x &= u\sin\theta_A - w\cos\theta_A \\ x &= \sin\theta_A\cos\theta_B\cos(\phi_B - \phi_A) - \cos\theta_A\sin\theta_B \end{aligned} \tag{8}$$

$$\begin{aligned} y &= v \\ y &= \cos\theta_B\sin(\phi_B - \phi_A) \end{aligned} \tag{9}$$

Using the above coordinates for point B and letting bearing be ω and distance be ρ, we may now calculate bearing and distance as follows,

$$\omega = \pi - \text{atan2}(y, x) \tag{10}$$

Note that atan2 is a function, found in the math library of the C programming language, that correctly determines the angle corresponding to (x, y), according to the conventions of the circular functions.

$$\rho = \frac{\pi}{2} - \tan^{-1}\frac{z}{\sqrt{x^2 + y^2}} \tag{11}$$

Notice that this is the distance in units of Earth radii. It would be useful to note that the radius of the Earth, in absolute terms, is 6371 km.